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Optimal estimates for the electric field in two dimensions

Habib Ammari [a,*], Hyeonbae Kang [b], Hyundae Lee [b], Jungwook Lee [b], Mikyoung Lim [a]

[a] *Centre de Mathématiques Appliquées, CNRS UMR 7641 and Ecole Polytechnique, 91128 Palaiseau Cedex, France*

[b] *School of Mathematical Sciences, Seoul National University, Seoul 151-747, South Korea*

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Abstract

We establish both upper and lower bounds on the electric field in the case where two circular conductivity inclusions are very close but not touching. We also obtain such bounds when a circular inclusion is very close to the boundary of a circular domain which contains the inclusion. The novelty of these estimates, which improve and make complete our earlier results in [H. Ammari, H. Kang, M. Lim, Gradient estimates for solutions to the conductivity problem, Math. Ann. 332 (2005) 277–286], is that they give an optimal information about the blow-up of the electric field as the conductivities of the inclusions degenerate.

Résumé

L'objet de cet article est d'établir des estimations précises sur le champ électrique dans des cas où deux objets conducteurs sont proches l'un de l'autre. La nouveauté de nos estimations est qu'elles donnent des bornes optimales, non seulement en fonction de la distance qui sépare les objets conducteurs, mais également en fonction de leurs conductivités dans les cas où elles dégénèrent.

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1. Introduction and statements of results

The purpose of this paper is to set out optimal gradient estimates for solutions to the isotropic conductivity problem in the presence of adjacent conductivity inclusions as the distance between the inclusions goes to zero and their conductivities degenerate. This difficult question arises in the study of composite media. Frequently in composites, the inclusions are very closely spaced and may even touch; see [9]. It is quite important from a practical point of view to know whether the electric field (the gradient of the potential) can be arbitrarily large as the inclusions get closer to each other or to the boundary of the background medium.

There have been some important works on the estimates of the gradient of the solution to the conductivity problem in the presence of inclusions. For finite and strictly positive conductivities, it was shown by Bonnetier and Vogelius

* Corresponding author.

E-mail addresses: ammari@cmapx.polytechnique.fr (H. Ammari), hkang@math.snu.ac.kr (H. Kang), hdlee@math.snu.ac.kr (H. Lee), wook9@snu.ac.kr (J. Lee), mklim@cmapx.polytechnique.fr (M. Lim).

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in [10] that the gradient of u remains bounded for circular touching inclusions of comparable radii. Li and Vogelius showed in [16] that ∇u is bounded independently of the distance between the inclusions B_1 and B_2, provided that the conductivities stay away from 0 and $+\infty$. It is worth mentioning that the result of [16] is much more general: it holds for arbitrary number of inclusions with arbitrary shape. This result has been recently extended to elliptic systems by Li and Nirenberg in [15]. On the other hand, for two identical perfectly conducting circular inclusions (with $k_1 = k_2 = +\infty$) which are ε apart, it has been shown in [8] (see also [17] and [12]) that the gradient generally becomes unbounded as the distance ε approaches zero. The rate at which this gradient becomes unbounded has actually been calculated in [8], for a special solution. For this special solution, the rate turns out to be $\varepsilon^{-1/2}$. In [6], a lower bound for the gradient of the solution to the conductivity problem for arbitrary conductivities, possibly degenerating, has been obtained. One of our objectives in this paper is to improve this bound and complete it by deriving an optimal upper one as well.

In this paper we consider the following two situations: when two circular conductivity inclusions are very close but not touching and when a circular inclusion is very close to the boundary of the domain where the inclusion is contained. These two simple two-dimensional models illustrate very well the feature of our estimates. We believe that they extend to arbitrary-shaped inclusions if their contact reduces to a point.

To describe the first situation we consider in this paper, let B_1 and B_2 be two circular inclusions contained in a matrix which we assume to be the free space $\mathbb{R}^2$. For $i = 1, 2$, we suppose that the conductivity k_i of the inclusion B_i is a constant different from the constant conductivity of the matrix, which is assumed to be 1 for convenience. The conductivity k_i of the inclusion may be 0 or $+\infty$. The zero conductivity indicates that the inclusion is an insulated inclusion while the infinite conductivity indicates a perfect conductor. We are especially interested in the case of extreme conductivities $k_i \to +\infty$ or $k_i \to 0$.

Given an entire harmonic function H, the first conductivity problem we consider in this paper is the following:

$$\begin{cases} \nabla \cdot \Big(1 + \sum_{i=1,2} (k_i - 1)\chi(B_i)\Big)\nabla u = 0 & \text{in } \mathbb{R}^2, \\ u(X) - H(X) = O(|X|^{-1}) & \text{as } |X| \to +\infty. \end{cases} \tag{1}$$

The electric field is given by ∇u, where u is the solution to (1) and represents the perturbation of the field ∇H in the presence of the two inclusions B_1 and B_2. For applications to the theory of composite materials, it is particularly important to consider the case when ∇H is a uniform field, i.e., $H(X) = A \cdot X$ for some constant vector A. Eq. (1) can be rewritten in the following form to emphasize the transmission conditions on ∂B_i, $i = 1, 2$:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus (\partial B_1 \cup \partial B_2), \\ u|_+ = u|_- & \text{on } \partial B_i,\ i = 1, 2, \\ \dfrac{\partial u}{\partial \nu}\Big|_+ = k_i \dfrac{\partial u}{\partial \nu}\Big|_- & \text{on } \partial B_i,\ i = 1, 2, \\ u(X) - H(X) = O(|X|^{-1}) & \text{as } |X| \to +\infty. \end{cases}$$

Here and throughout this paper the subscript $\pm$ indicates the limit from outside and inside the domain, respectively. If $k_i = 0$, then the transmission condition on the normal derivatives of u should be replaced with $\frac{\partial u}{\partial \nu}|_+ = 0$ on ∂B_i, while if $k_i = +\infty$, it should be replaced with $u = \text{constant}$ on B_i.

As has already been said, we are interested in the behavior of the gradient of the solution to Eq. (1) as the distance between the inclusions B_1 and B_2 goes to zero for arbitrary conductivities k_1 and k_2, possibly degenerating.

To state the first main result of this paper, let us first fix notation. For $i = 1, 2$, let $B_i = B(Z_i, r_i)$, the disk centered at Z_i and of radius r_i. Let R_i, $i = 1, 2$, be the reflection with respect to ∂B_i, i.e.,

$$R_i(X) := \frac{r_i^2(X - Z_i)}{|X - Z_i|^2} + Z_i, \quad i = 1, 2.$$

It is easy to see that the combined reflection R_1R_2 and R_2R_1 have unique fixed points. Let I be the line segment between these two fixed points. Let X_j, $j = 1, 2$, be the point on ∂B_j closest to the other disk. We also let:

$$r_{\min} := \min(r_1, r_2), \quad r_{\max} := \max(r_1, r_2), \quad r_* := \sqrt{(2r_1r_2)/(r_1 + r_2)},$$
$$\lambda_i := \frac{k_i + 1}{2(k_i - 1)}, \quad i = 1, 2 \quad \text{and} \quad \tau := \frac{1}{4\lambda_1\lambda_2}.$$

We obtain the following result on the blow-up of the gradient.

Theorem 1.1. *Let $\varepsilon := \operatorname{dist}(B_1, B_2)$ and let $\nu^{(j)}$ and $T^{(j)}$, $j = 1, 2$, be the unit normal and tangential vector fields to ∂B_j, respectively. Let u be the solution of* (1).

(i) *If ε is sufficiently small, there is a constant C_1 independent of k_1, k_2, r_1, r_2, and ε such that*

$$\frac{C_1 \inf_{X\in I} |\langle \nabla H(X), \nu^{(j)}(X_j)\rangle|}{1-\tau+(r_*/r_{\min})\sqrt{\varepsilon}} \leqslant \big|\nabla u|_+(X_j)\big|, \quad j=1,2, \tag{2}$$

provided that $k_1, k_2 > 1$, and

$$\frac{C_1 \inf_{X\in I} |\langle \nabla H(X), T^{(j)}(X_j)\rangle|}{1-\tau+(r_*/r_{\min})\sqrt{\varepsilon}} \leqslant \big|\nabla u|_+(X_j)\big|, \quad j=1,2, \tag{3}$$

provided that $k_1, k_2 < 1$.

(ii) *Let Ω be a bounded set containing B_1 and B_2. Then there is a constant C_2 independent of k_1, k_2, r_1, r_2, ε, and Ω such that*

$$\|\nabla u\|_{L^\infty(\Omega)} \leqslant \frac{C_2\|\nabla H\|_{L^\infty(\Omega)}}{1-|\tau|+(r_*/r_{\max})\sqrt{\varepsilon}}. \tag{4}$$

Note that if $H(X) = A\cdot X$ for some constant vector A, which is the most interesting case, then the quantity

$$\big\langle \nabla H(X), \nu^{(j)}(X_j)\big\rangle = \big\langle A, \nu^{(j)}(X_j)\big\rangle,$$

and hence it does not vanish if we choose A appropriately.

Theorem 1.1 quantifies the behavior of ∇u in terms of the conductivities of the inclusions, their radii, and the distance between them. For example, if k_1 and k_2 degenerate to $+\infty$ or zero, then $\tau = 1$ and hence (2) and (4) read:

$$\frac{C_1'}{(r_*/r_{\min})\sqrt{\varepsilon}} \leqslant \big|\nabla u(X_j)\big|, \quad j=1,2, \ \ \|\nabla u\|_{L^\infty(\Omega)} \leqslant \frac{C_2'}{(r_*/r_{\max})\sqrt{\varepsilon}}, \tag{5}$$

for some positive constants C_1' and C_2', which shows that ∇u blows up at the rate of $\varepsilon^{-1/2}$ as the inclusions get closer. It further shows that the gradient blows up at X_1 and X_2, X_j for $j = 1, 2$, being the point on ∂B_j closest to the other disk.

The lower bounds in (2) and (3) are improved versions of the one obtained in [6]. The proofs of our new estimates make use of quite explicit but nontrivial expansion formulae, originally derived in [5]. They are achieved by using a significantly different method from [10,16,8].

Another interesting situation is when the inclusion is very close to the boundary. To describe this second situation, suppose that Ω, which is a disk of radius ρ, contains an inclusion B, which is a disk of radius r. Suppose also that the conductivity of Ω is 1 and that of B is $k \neq 1$. The conductivity problems considered in this case are the following Dirichlet and Neumann problems: for a given $f \in \mathcal{C}^{1,\alpha}(\partial\Omega)$, $\alpha > 0$,

$$\begin{cases} \nabla\cdot(1+(k-1)\chi(B))\nabla u = 0 & \text{in } \Omega,\\ u = f & \text{on } \partial\Omega, \end{cases} \tag{6}$$

and for a given $g \in \mathcal{C}^{\alpha}(\partial\Omega)$,

$$\begin{cases} \nabla\cdot(1+(k-1)\chi(B))\nabla u = 0 & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases} \tag{7}$$

To ensure existence and uniqueness of a solution to (7), we suppose that $\int_{\partial\Omega} g = 0$ and $\int_{\partial\Omega} u = 0$. Let X_1 be the point on ∂B closest to $\partial\Omega$ and X_2 be the point on $\partial\Omega$ closest to ∂B, and let R_B and R_Ω are reflections with respect to ∂B and $\partial\Omega$, respectively. Let P_1 and P_2 be fixed points of $R_B R_\Omega$ and $R_\Omega R_B$, respectively, and let J_1 be the line segment between P_1 and X_1 and J_2 that between P_2 and X_2.

The second main result of this paper is the following triplet of estimates for the gradient of the solutions to (6) and (7). Let $\mathcal{D}_\Omega(f)$ and $\mathcal{S}_\Omega(g)$ denote the double and single layer potentials whose definitions are given in Section 2. For the Dirichlet problem we have the following theorem:

Theorem 1.2. *Let*

$$\varepsilon := \operatorname{dist}(B, \partial\Omega), \qquad \sigma := \frac{k-1}{k+1}, \qquad r^* := \sqrt{\frac{\rho - r}{\rho r}},$$

and let u be the solution to (6).

(i) *If $k > 1$, then there exists a constant C_1 independent of k, r, ε, and f such that for ε small enough,*

$$\frac{C_1 \inf_{X \in J_1} |\langle \nabla \mathcal{D}_\Omega(f)(X), \nu_B(X_1)\rangle|}{1 - \sigma + 4r^*\sqrt{\varepsilon}} \leqslant \big|\nabla u|_+(X_1)\big|, \tag{8}$$

and

$$\frac{C_1 \inf_{X \in J_2} |\langle \nabla \mathcal{D}_\Omega(f)(X), \nu_\Omega(X_2)\rangle|}{1 - \sigma + 4r^*\sqrt{\varepsilon}} \leqslant \big|\nabla u|_-(X_2)\big|. \tag{9}$$

Here ν_B and ν_Ω denote the outward unit normal to ∂B and $\partial\Omega$.

(ii) *For any $k \neq 1$, there exists a constant C_2 independent of k, r, and ε such that for ε small enough,*

$$\|\nabla u\|_{L^\infty(\Omega)} \leqslant \frac{C_2 \|f\|_{\mathcal{C}^{1,\alpha}(\partial\Omega)}}{1 - |\sigma| + r^*\sqrt{\varepsilon}}. \tag{10}$$

For the Neumann problem the following theorem holds:

Theorem 1.3. *Let ε, σ, r^* be defined as in Theorem* 1.2.

(i) *If $k < 1$, then there exists a constant C_1 independent of k, r, ε, and g such that for ε small enough,*

$$\frac{C_1 \inf_{X \in J_1} |\langle \nabla \mathcal{S}_\Omega(g)(X), T_B(X_1)\rangle|}{1 + \sigma + 4r^*\sqrt{\varepsilon}} \leqslant \big|\nabla u|_+(X_1)\big|, \tag{11}$$

and

$$\frac{C_1 \inf_{X \in J_2} |\langle \nabla \mathcal{S}_\Omega(g)(X), T_\Omega(X_2)\rangle|}{1 + \sigma + 4r^*\sqrt{\varepsilon}} \leqslant \big|\nabla u|_-(X_2)\big|. \tag{12}$$

Here T_B and T_Ω denote the positively oriented unit tangent vector fields on ∂B and $\partial\Omega$, respectively.

(ii) *For any $k \neq 1$, there exists a constant C_2 independent of k, r, and ε such that for ε small enough,*

$$\|\nabla u\|_{L^\infty(\Omega)} \leqslant \frac{C_2 \|g\|_{\mathcal{C}^{\alpha}(\partial\Omega)}}{1 - |\sigma| + r^*\sqrt{\varepsilon}}. \tag{13}$$

If Z is the center of Ω and if $f(X) = A \cdot X$ for some constant vector A, then $\mathcal{D}_\Omega(f)(X) = \frac{1}{2} A \cdot X$ for $X \in \Omega$ and $\mathcal{D}_\Omega(f)(X) = -\frac{\rho A \cdot X}{2|X - Z|^2}$ for $X \in \mathbb{R}^2 \setminus \overline{\Omega}$, and hence we can achieve:

$$\big\langle \nabla \mathcal{D}_\Omega(f)(X), \nu_B(X_1)\big\rangle \neq 0 \quad \text{and} \quad \big\langle \nabla \mathcal{D}_\Omega(f)(X), \nu_\Omega(X_2)\big\rangle \neq 0 \quad \text{for any } X,$$

by choosing A appropriately. Likewise, if $g := A \cdot \nu$ on $\partial\Omega$, then $\mathcal{S}_\Omega(g) = -\frac{1}{2} A \cdot X + \text{constant}$.

Theorem 1.2 shows that in the case of the Dirichlet problem, if the inclusion is a perfect conductor ($k = +\infty$ and hence $\sigma = 1$), then

$$\frac{C_1'}{r^*\sqrt{\varepsilon}} \leqslant \|\nabla u\|_{L^\infty(\Omega)} \leqslant \frac{C_2'}{r^*\sqrt{\varepsilon}},$$

for some positive constants C_1' and C_2'. Thus ∇u blows up at the rate of $\varepsilon^{-1/2}$ as long as the magnitude of r is much larger than that of ε. It also shows that the gradient blows up at the points X_1 and X_2. On the other hand, for the Neumann problem, according to Theorem 1.3 the situation is reversed: ∇u blows up for an insulator. If r is of the same order as ε, then $r^* \approx \frac{1}{\sqrt{\varepsilon}}$ and hence ∇u does not blow up. In fact, it stays bounded and an asymptotic expansion of the solution as $\varepsilon \to 0$ can be derived. See for instance [1,4,7] for this.

One may think that Theorems 1.2 and 1.3 can be easily derived from Theorem 1.1 by reflection or conformal mapping. This is, as it will be shown, far from being true. The proofs of Theorems 1.2 and 1.3 require delicate analysis and careful and tricky estimates.

In this paper we only deal with the two dimensional case. It seems challenging to obtain similar results in three dimensions. At this moment it is even not clear what the blow-up rate of the gradient would be in three dimensions. In this direction, we have recently established in [2] that, unlike the two-dimensional case, if the inclusions are grounded conductors (the Dirichlet boundary condition on the inclusions is set to be zero) and of spherical shape, the gradient stays, to our surprise, bounded regardless of the separation distance between them.

This paper is organized as follows. In Section 2 we review some preliminary facts on layer potentials and representations of the solution to the conductivity problem obtained in [6]. Theorem 1.1 is proved in Section 3, and Theorems 1.2 and 1.3 in Section 4. Although our results hold for special cases, we believe that they extend to arbitrary-shaped inclusions if their contact reduces to a point.

2. Preliminaries

To make our paper self-contained and our exposition clear, we introduce our main tools for studying the conductivity problems and collect some preliminary results regarding layer potentials. We also linger over a description of some results from our earlier papers. The material in Lemmas 2.5 and 2.6 is however, up to our knowledge, new.

Let D be a bounded Lipschitz domain in $\mathbb{R}^2$ and $\mathcal{S}_D\phi$ and $\mathcal{D}_D\phi$ denote the single and double layer potentials of a function $\phi \in L^2(\partial D)$, namely,

$$\mathcal{S}_D\phi(X) = \frac{1}{2\pi}\int_{\partial D} \ln|X-Y|\phi(Y)\,\mathrm{d}\sigma(Y), \quad X \in \mathbb{R}^2,$$

$$\mathcal{D}_D\phi(X) = \frac{1}{2\pi}\int_{\partial D} \frac{\langle Y-X, \nu_Y\rangle}{|X-Y|^2}\phi(Y)\,\mathrm{d}\sigma(Y), \quad X \in \mathbb{R}^2 \setminus \partial D.$$

For a function u defined on $\mathbb{R}^2 \setminus \partial D$, we set:

$$\frac{\partial u}{\partial \nu}\bigg|_{\pm}(X) := \lim_{t\to 0^+}\big\langle \nabla u(X \pm t\nu_X), \nu_X\big\rangle, \quad X \in \partial D,$$

if the limits exist. Here and throughout this paper ν_X is the outward normal to ∂D at X and $\langle\,,\rangle$ denotes the scalar product in $\mathbb{R}^2$. The following jump relations for the single and double layer potentials are well known [11,18,3].

Lemma 2.1. *For $\phi \in L^2(\partial D)$ we have*:

$$\frac{\partial}{\partial\nu}\mathcal{S}_D\phi|_{\pm}(X) = \left(\pm\frac{1}{2} + \mathcal{K}_D^*\right)\phi(X) \quad \text{a.e. } X \in \partial D, \tag{14}$$

$$(\mathcal{D}_D\phi)|_{\pm}(X) = \left(\mp\frac{1}{2} + \mathcal{K}_D\right)\phi(X) \quad \text{a.e. } X \in \partial D, \tag{15}$$

where $\mathcal{K}_D$ is defined by:

$$\mathcal{K}_D\phi(X) = \frac{1}{2\pi}\,\mathrm{p.v.}\int_{\partial D} \frac{\langle Y-X, \nu_Y\rangle}{|X-Y|^2}\phi(Y)\,\mathrm{d}\sigma(Y),$$

and $\mathcal{K}_D^*$ is the L^2-adjoint of $\mathcal{K}_D$, i.e.,

$$\mathcal{K}_D^*\phi(X) = \frac{1}{2\pi}\,\mathrm{p.v.}\int_{\partial D} \frac{\langle X-Y, \nu_X\rangle}{|X-Y|^2}\phi(Y)\,\mathrm{d}\sigma(Y).$$

Here p.v. *denotes the Cauchy principal value.*

Note that if D is a two-dimensional disk with radius r, then

$$\frac{\langle X-Y, \nu_X\rangle}{|X-Y|^2} = \frac{1}{2r}, \quad \forall X, Y \in \partial D,\ X \neq Y,$$

and hence

$$\mathcal{K}_D^* \phi(X) = \mathcal{K}_D \phi(X) = \frac{1}{4\pi r} \int_{\partial D} \phi(Y)\,\mathrm{d}\sigma, \quad X \in \partial D. \tag{16}$$

When D is a disk, we define the function $R_D(f)$ for a function f by:

$$R_D(f)(X) := f\big(R_D(X)\big).$$

Using the jump formula for the single layer potential, we get the following lemma [5].

Lemma 2.2. *Let D be a disk in $\mathbb{R}^2$ and let R_D denote the reflection with respect to ∂D. If v is a harmonic in D and continuous on $\overline{D}$, then*

$$\mathcal{S}_D\left(\frac{\partial v}{\partial \nu}\bigg|_{\partial D}\right)(X) = -\frac{1}{2}(R_D v)(X) + C, \quad X \in \mathbb{R}^2 \setminus \overline{D}, \tag{17}$$

where C is some constant. Analogously, if v is harmonic in $\mathbb{R}^2 \setminus \overline{D}$, continuous on $\mathbb{R}^2 \setminus D$, and $v(X) \to 0$ as $|X| \to 0$, then

$$\mathcal{S}_D\left(\frac{\partial v}{\partial \nu}\bigg|_{\partial D}\right)(X) = \frac{1}{2}(R_D v)(X) + C, \quad X \in \overline{D}, \tag{18}$$

for some constant C.

We need the following lemma which was first proved in [6].

Lemma 2.3. *Suppose that B_1 and B_2 are two disjoint disks and let $R_i = R_{B_i}$ be the reflection with respect to ∂B_i. Then the solution to* (1) *is represented as*

$$u(X) = H(X) + \mathcal{S}_{B_1}\varphi_1(X) + \mathcal{S}_{B_2}\varphi_2(X), \quad X \in \Omega, \tag{19}$$

where $\varphi_i \in L_0^2(\partial B_i)$, $i = 1, 2$, is the unique solution to the system of integral equations:

$$\lambda_l \varphi_l - \frac{\partial(\mathcal{S}_{B_i}\varphi_i)}{\partial \nu^{(l)}}\bigg|_{\partial B_i} = \frac{\partial H}{\partial \nu}\bigg|_{\partial B_i} \quad \text{on } \partial B_l,\ l = 1, 2,\ i \neq l, \tag{20}$$

with $\lambda_i = \frac{k_i+1}{2(k_i-1)}$ and $\nu^{(l)}$ is the outward unit normal to ∂B_l. Moreover, the potentials φ_1 and φ_2 are explicitly given by:

$$\begin{aligned}
\varphi_1 &= \frac{1}{\lambda_1}\sum_{m=0}^{+\infty} \frac{1}{(4\lambda_1\lambda_2)^m} \frac{\partial}{\partial \nu^{(1)}}\left[(R_2R_1)^m\left(I - \frac{1}{2\lambda_2}R_2\right)H\right]\bigg|_{\partial B_1}, \\
\varphi_2 &= \frac{1}{\lambda_2}\sum_{m=0}^{+\infty} \frac{1}{(4\lambda_1\lambda_2)^m} \frac{\partial}{\partial \nu^{(2)}}\left[(R_1R_2)^m\left(I - \frac{1}{2\lambda_1}R_1\right)H\right]\bigg|_{\partial B_2},
\end{aligned} \tag{21}$$

where the series in (21) *converge absolutely and uniformly.*

In Lemma 2.3, the space $L_0^2(\partial B_i)$ denotes the set of all $g \in L^2(\partial B_i)$ having mean value zero: $\int_{\partial B_i} g = 0$. The following lemma from [6] is also of use to us.

Lemma 2.4. *Let u be the solution of* (1) *and let $\widetilde{H}$ be a harmonic conjugate to H. Let v be the solution to the conductivity problem*:

$$\begin{cases} \nabla \cdot \left(1 + \sum_{i=1,2}\left(\frac{1}{k_i} - 1\right)\chi(B_i)\right)\nabla v = 0 & \text{in } \mathbb{R}^2, \\ v(X) - \widetilde{H}(X) = O(|X|^{-1}). \end{cases} \tag{22}$$

Then

$$\frac{\partial u}{\partial T} = -\frac{\partial v}{\partial \nu^{(i)}}\bigg|_+ \quad \text{on } \partial B_i,\ i = 1, 2.$$

We now turn our attention to the second situation, i.e. problems (6) and (7), when both Ω and B are disks. We first note that $\mathcal{D}_\Omega f$ is $\mathcal{C}^{1,\alpha}$ in $\overline{\Omega}$ and $\mathbb{R}^2 \setminus \Omega$ since $f \in \mathcal{C}^{1,\alpha}(\partial\Omega)$. It was shown in [13,14] that the solution u to the problem (6) for a fixed Dirichlet data f is given by:

$$u(X) = \mathcal{D}_\Omega(f)(X) - \mathcal{S}_\Omega(g)(X) + \mathcal{S}_B(\varphi)(X), \qquad X \in \Omega, \qquad g := \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega}, \tag{23}$$

where φ with mean value zero satisfies the integral equation,

$$(\lambda I - \mathcal{K}_B^*)\varphi = \frac{\partial}{\partial \nu}\big(\mathcal{D}_\Omega(f) - \mathcal{S}_\Omega(g)\big) \quad \text{on } \partial B,$$

with $\lambda = \frac{k+1}{2(k-1)}$. Since B is a disk, it follows from (16) that $\mathcal{K}_B^*\varphi \equiv 0$ on $L_0^2(\partial B)$ and hence

$$\lambda\varphi = \frac{\partial}{\partial \nu}\big(\mathcal{D}_\Omega(f) - \mathcal{S}_\Omega(g)\big) \quad \text{on } \partial B. \tag{24}$$

On the other hand, $g = \frac{\partial u}{\partial \nu}|_{\partial\Omega}$ yields:

$$g = \frac{\partial}{\partial \nu}\big(\mathcal{D}_\Omega(f) - \mathcal{S}_\Omega(g) + \mathcal{S}_B(\varphi)\big)\big|_- \quad \text{on } \partial\Omega.$$

Since $\frac{\partial}{\partial \nu}\mathcal{S}_\Omega(g)|_- = (-\frac{1}{2}I + \mathcal{K}_\Omega^*)g$ and Ω is a disk, $\frac{\partial}{\partial \nu}\mathcal{S}_\Omega(g)|_- = -\frac{1}{2}g$ on ∂B. Thus we get:

$$\frac{1}{2}g = \frac{\partial}{\partial \nu}\big(\mathcal{D}_\Omega(f) + \mathcal{S}_B(\varphi)\big)\big|_- \quad \text{on } \partial\Omega. \tag{25}$$

It then follows from (24) and (25) that g and φ are the solution of the following system of integral equations:

$$\begin{cases} \dfrac{1}{2}g - \dfrac{\partial(\mathcal{S}_B\varphi)}{\partial \nu_\Omega} = \dfrac{\partial(\mathcal{D}_\Omega f)}{\partial \nu_\Omega} & \text{on } \partial\Omega, \\ \lambda\varphi + \dfrac{\partial(\mathcal{S}_\Omega g)}{\partial \nu_B} = \dfrac{\partial(\mathcal{D}_\Omega f)}{\partial \nu_B} & \text{on } \partial B. \end{cases} \tag{26}$$

Observe the similarity of (26) to (20). Using the same argument as the one introduced in deriving (21), one can show that the following lemma holds.

Lemma 2.5. *Let g and φ be the functions given in* (23). *Then g and φ are given by*:

$$\begin{aligned} g &= 2\sum_{m=0}^{+\infty} \frac{1}{(2\lambda)^m}\frac{\partial}{\partial \nu_\Omega}\bigg[(R_B R_\Omega)^m\bigg(I - \frac{1}{2\lambda}R_B\bigg)\mathcal{D}_\Omega f\bigg] \quad \text{on } \partial\Omega, \\ \varphi &= \frac{1}{\lambda}\sum_{m=0}^{+\infty} \frac{1}{(2\lambda)^m}\frac{\partial}{\partial \nu_B}\big[(R_\Omega R_B)^m (I - R_\Omega)\mathcal{D}_\Omega f\big] \quad \text{on } \partial B, \end{aligned} \tag{27}$$

where the series in (27) *converge absolutely and uniformly.*

Proof of Lemma 2.5. The convergence of the formula (27) will be proved in the course of proving Theorem 1.2 in Section 4.

We first prove that for $(h_1, h_2) \in L_0^2(\partial\Omega) \times L_0^2(\partial B)$ there exists a unique solution $(g, \varphi) \in L_0^2(\partial\Omega) \times L_0^2(\partial B)$ such that

$$\begin{cases} \dfrac{1}{2}g - \dfrac{\partial(\mathcal{S}_B\varphi)}{\partial \nu_\Omega} = h_1 & \text{on } \partial\Omega, \\ \lambda\varphi + \dfrac{\partial(\mathcal{S}_\Omega g)}{\partial \nu_B} = h_2 & \text{on } \partial B. \end{cases}$$

Since B is away from $\partial\Omega$, the operator $(g,\varphi) \to (-\frac{\partial(\mathcal{S}_B\varphi)}{\partial\nu_\Omega}, \frac{\partial(\mathcal{S}_\Omega g)}{\partial\nu_B})$ is a compact operator on $L^2_0(\partial\Omega) \times L^2_0(\partial B)$. Thus, by the Fredholm alternative, it suffices to prove that if $(h_1,h_2) = (0,0)$, then the solution $(g,\varphi) = (0,0)$. In order to do this, suppose that

$$\begin{cases} \dfrac{1}{2}g - \dfrac{\partial(\mathcal{S}_B\varphi)}{\partial\nu_\Omega} = 0 & \text{on } \partial\Omega, \\ \lambda\varphi + \dfrac{\partial(\mathcal{S}_\Omega g)}{\partial\nu_B} = 0 & \text{on } \partial B. \end{cases}$$

Then the function $u = \mathcal{S}_\Omega(g) + \mathcal{S}_B(\varphi)$ in Ω is a solution of $\nabla \cdot (1 + (\frac{1}{k} - 1)\chi(B))\nabla u = 0$ in Ω and satisfies $\frac{\partial u}{\partial\nu}|_- = 0$ on $\partial\Omega$. This implies that u is constant in Ω, and hence $\mathcal{S}_B(\varphi)$ is harmonic in Ω. It then follows from the jump formula (14) that $\varphi = 0$ and therefore, $\mathcal{S}_\Omega(g) = \text{constant}$ in Ω. Hence, $g = -2\frac{\partial\mathcal{S}_\Omega g}{\partial\nu}|_- = 0$.

We now prove that the pair (g,φ) given by (27) satisfies (26). Observe that the function,

$$(R_B R_\Omega)^m\left(I - \frac{1}{2\lambda}R_B\right)(\mathcal{D}_\Omega f)(X) = (\mathcal{D}_\Omega f)\big((R_\Omega R_B)^m(X)\big) - \frac{1}{2\lambda}(\mathcal{D}_\Omega f)\big(R_B(R_\Omega R_B)^m(X)\big),$$

is harmonic in $\mathbb{R}^2 \setminus \overline{B}$ and approaches to,

$$(\mathcal{D}_\Omega f)\big((R_\Omega R_B)^{m-1}R_\Omega(Z)\big) - \frac{1}{2\lambda}(\mathcal{D}_\Omega f)\big(R_B(R_\Omega R_B)^{m-1}R_\Omega(Z)\big),$$

as $|X| \to +\infty$, where Z is the center of B. Since $\mathcal{S}_\Omega(1)$ is constant in Ω, it follows from (17) that

$$\begin{aligned} \frac{\partial(\mathcal{S}_\Omega g)}{\partial\nu_B} &= \sum_{m=0}^{+\infty}\frac{1}{(2\lambda)^m}\frac{\partial}{\partial\nu_B}\left[R_\Omega(R_B R_\Omega)^m\left(I - \frac{1}{2\lambda}R_B\right)\mathcal{D}_\Omega f\right] \\ &= \sum_{m=0}^{+\infty}\frac{1}{(2\lambda)^m}\frac{\partial}{\partial\nu_B}\big[(R_\Omega R_B)^m(R_\Omega - I)\mathcal{D}_\Omega f\big] + \frac{\partial(\mathcal{D}_\Omega f)}{\partial\nu_B}. \end{aligned}$$

Likewise one can show that

$$\frac{\partial(\mathcal{S}_B\varphi)}{\partial\nu_\Omega} = \sum_{m=0}^{+\infty}\frac{1}{(2\lambda)^m}\frac{\partial}{\partial\nu_\Omega}\left[(R_B R_\Omega)^m\left(I - \frac{1}{2\lambda}R_B\right)\mathcal{D}_\Omega f\right] - \frac{\partial(\mathcal{D}_\Omega f)}{\partial\nu_\Omega}.$$

Thus (g,φ) satisfies (26) and the proof is complete. □

The representation of g and φ given in (27) can be simplified using the relation

$$R_\Omega\mathcal{D}_\Omega f(X) = \mathcal{D}_\Omega f\big(R_\Omega(X)\big) = -\mathcal{D}_\Omega f(X) + \text{constant}, \qquad X \in \mathbb{R}^2 \setminus \partial\Omega, \tag{28}$$

which follows from (15) and (16) since Ω is a disk. Using (28), we then compute:

$$\begin{aligned} g &= 2\sum_{m=0}^{+\infty}\frac{1}{(2\lambda)^m}\frac{\partial}{\partial\nu_\Omega}\left[(R_B R_\Omega)^m\left(I - \frac{1}{2\lambda}R_B\right)\mathcal{D}_\Omega f\right] \\ &= 2\sum_{m=0}^{+\infty}\frac{1}{(2\lambda)^m}\frac{\partial}{\partial\nu_\Omega}\left[(R_B R_\Omega)^m\left(I + \frac{1}{2\lambda}R_B R_\Omega\right)\mathcal{D}_\Omega f\right] \\ &= 4\sum_{m=1}^{+\infty}\frac{1}{(2\lambda)^m}\frac{\partial}{\partial\nu_\Omega}\big[(R_B R_\Omega)^m\mathcal{D}_\Omega f\big] + 2\frac{\partial}{\partial\nu_\Omega}\mathcal{D}_\Omega f \quad \text{on } \partial\Omega. \end{aligned} \tag{29}$$

Likewise we can show that

$$\varphi = \frac{2}{\lambda}\sum_{m=0}^{+\infty}\frac{1}{(2\lambda)^m}\frac{\partial}{\partial\nu_B}\big[(R_\Omega R_B)^m\mathcal{D}_\Omega f\big] \quad \text{on } \partial B. \tag{30}$$

The following lemma is also of importance to us.

Lemma 2.6. *Let u be the solution of* (7) *for $g \in \mathcal{C}^\alpha(\partial\Omega)$ and let G be the function satisfying $\frac{\partial G}{\partial T} = g$ on $\partial\Omega$ and $\int_{\partial\Omega} G = 0$. Define v to be the solution of the following conductivity problem*:

$$\begin{cases} \nabla \cdot \left(1 + \left(\frac{1}{k} - 1\right)\chi(B)\right)\nabla v = 0 & \text{in } \Omega, \\ v = G & \text{on } \partial\Omega. \end{cases} \tag{31}$$

Then

$$\frac{\partial u}{\partial T} = -\frac{\partial v}{\partial \nu}\bigg|_+ \quad \text{on } \partial B. \tag{32}$$

Moreover, $\mathcal{D}_\Omega(v|_{\partial\Omega})$ is a harmonic conjugate to $\mathcal{S}_\Omega g$ in Ω.

Proof. Let w be a harmonic conjugate of u in $\Omega \setminus \overline{B}$ and B. Such a conjugate function exists in $\Omega \setminus \overline{B}$ since $\int_C \frac{\partial u}{\partial \nu}\, d\sigma = 0$ for any simple closed curve C in $\Omega \setminus \overline{B}$. Moreover, since u is $\mathcal{C}^{1,\alpha}$, so is w.

Define v by

$$v(X) := \begin{cases} w(X), & X \in \Omega \setminus \overline{B}, \\ kw(X) - \dfrac{k}{|\partial B|}\displaystyle\int_{\partial B} w \, d\sigma, & X \in B. \end{cases} \tag{33}$$

Then one can see from the Cauchy–Riemann equation and the transmission conditions on u that

$$\frac{\partial v}{\partial T}\bigg|_+ = \frac{\partial u}{\partial \nu}\bigg|_+ = k\frac{\partial u}{\partial \nu}\bigg|_- = \frac{\partial v}{\partial T}\bigg|_-,$$
$$\frac{\partial v}{\partial \nu}\bigg|_+ = -\frac{\partial u}{\partial T}\bigg|_+ = -\frac{\partial u}{\partial T}\bigg|_- = \frac{1}{k}\frac{\partial v}{\partial \nu}\bigg|_-.$$

Thus v defined by (33) is the unique solution to (22), and hence (32) holds.

It follows from (15) that

$$\mathcal{D}_\Omega(v|_{\partial\Omega})|_- = \frac{1}{2}v \quad \text{on } \partial\Omega,$$

and hence

$$\frac{\partial(\mathcal{D}_\Omega(v|_{\partial\Omega}))}{\partial T} = \frac{1}{2}\frac{\partial v}{\partial T} = \frac{1}{2}g = -\frac{\partial(\mathcal{S}_\Omega g)}{\partial \nu} \quad \text{on } \partial\Omega.$$

Therefore $\mathcal{D}_\Omega(v|_{\partial\Omega})$ is a harmonic conjugate of $\mathcal{S}_\Omega g$ in Ω. This completes the proof. □

3. Proof of Theorem 1.1

At this point we have all the necessary ingredients to prove Theorem 1.1. As has been said, the lower bound in (2) and (3) is an improvement of the one obtained in [6].

Recall that there are two disks $B_j = B(Z_j, r_j)$, $j = 1, 2$, inside Ω, and that R_j is the reflection with respect to ∂B_j. We suppose that both centers Z_1 and Z_2 are on the x-axis.

If X is on the x-axis, i.e., $X = (x, 0)$, straightforward calculations show that

$$DR_i(X) = g_i(X)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\nabla(R_i f)(X) = \nabla f\big(R_i(X)\big)g_i(X)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{34}$$

where

$$g_i(X) = \frac{r_i^2}{|X - Z_i|^2}, \quad i = 1, 2. \tag{35}$$

Therefore,

$$\nabla\big((R_2R_1)^m H\big)(X) = \left[\prod_{i=1}^{2m} g_{l_i}\big(R_{l_{i-1}}\cdots R_{l_1}(X)\big)\right]\nabla H\big((R_1R_2)^m(X)\big)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{36}$$

and

$$\nabla\big((R_2R_1)^m R_2 H\big)(X) = g_2\big((R_1R_2)^m(X)\big)\left[\prod_{i=1}^{2m} g_{l_i}\big(R_{l_{i-1}}\cdots R_{l_1}(X)\big)\right]\nabla H\big(R_2(R_1R_2)^m(X)\big)\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \tag{37}$$

where $(l_1,\dots,l_{2m}) = (\overbrace{2,1,2,1,\dots,2,1}^{m\text{ times}})$.

Let u be the solution to Eq. (1). Combining (14), (16), (19), and (20) yields:

$$\frac{\partial u}{\partial \nu^{(i)}}\bigg|_{\pm} = \frac{\partial H}{\partial \nu^{(i)}} + \frac{\partial(\mathcal{S}_{B_2}\varphi_2)}{\partial \nu^{(i)}}\bigg|_{\pm} + \frac{\partial(\mathcal{S}_{B_1}\varphi_1)}{\partial \nu^{(i)}}\bigg|_{\pm} = \left(\lambda_i \pm \frac{1}{2}\right)\varphi_i \quad \text{on } \partial B_i,\ i=1,2.$$

Consequently,

$$\big|\nabla u|_{\pm}(X_i)\big| \geqslant \left|\frac{\partial u}{\partial \nu^{(i)}}\bigg|_{\pm}(X_i)\right| \geqslant \left|\lambda_i \pm \frac{1}{2}\right|\big|\varphi_i(X_i)\big|. \tag{38}$$

Suppose that $k_1 > 1$ and $k_2 > 1$. By (21), (36) and (37), we obtain the following inequality:

$$\begin{aligned}\big|\varphi_1(X_1)\big| &\geqslant \frac{1}{\lambda_1}\sum_{m=0}^{+\infty}\frac{1}{(4\lambda_1\lambda_2)^m}\left(a^{2m} + \frac{1}{2\lambda_2}a^{2m+1}\right)\inf_I\big|\nabla H\cdot \nu^{(1)}\big|\\ &\geqslant \frac{1}{\lambda_1}\frac{1+\frac{a}{2\lambda_2}}{1-\frac{a^2}{4\lambda_1\lambda_2}}\inf_I\big|\nabla H\cdot\nu^{(1)}\big|,\end{aligned}$$

where $a := (1+2(r_*/r_{\min})\sqrt{\varepsilon})^{-1}$. We then get from (38) that

$$\big|\nabla u|_+(X_1)\big| \geqslant \frac{C}{1-\tau+(r_*/r_{\min})\sqrt{\varepsilon}}\inf_I\big|\nabla H\cdot\nu^{(1)}\big|.$$

Likewise we obtain:

$$\big|\nabla u|_+(X_2)\big| \geqslant \frac{C}{1-\tau+(r_*/r_{\min})\sqrt{\varepsilon}}\inf_I\big|\nabla H\cdot\nu^{(2)}\big|.$$

Thus the lower bound in (2) is now derived.

If both k_1 and k_2 are less than 1, then one can use Lemma 2.4 to obtain the lower bound in (3). See [6] for the details.

We now prove (4). We need the following lemmas:

Lemma 3.1. *Let $r_{\max}, r_{\min}, r_*$, and ε be as in Theorem* 1.1. *If ε is small enough, then for any $X \in \overline{B}_1$ and $n \geqslant 8r_*/\sqrt{\varepsilon}$ we have*:

$$\big|(R_1R_2)^n(X) - Z_2\big| \geqslant r_2\left(1+\frac{r_*}{2r_{\max}}\sqrt{\varepsilon}\right),$$

and

$$\big|(R_2R_1)^n R_2(X) - Z_1\big| \geqslant r_1\left(1+\frac{r_*}{2r_{\max}}\sqrt{\varepsilon}\right).$$

For any $X \in \overline{B}_2$ and $n \geqslant 8r_*/\sqrt{\varepsilon}$ we have:

$$\big|(R_2R_1)^n(X) - Z_1\big| \geqslant r_1\left(1+\frac{r_*}{2r_{\max}}\sqrt{\varepsilon}\right),$$

and

$$\left|(R_1R_2)^n R_1(X) - Z_2\right| \geqslant r_2\left(1 + \frac{r_*}{2r_{\max}}\sqrt{\varepsilon}\right).$$

Proof. After a translation and a rotation if necessary, we may assume that $B_1 = B((0,0), r_1)$ and $B_2 = B((r_1+r_2+\varepsilon, 0), r_2)$, i.e., $Z_1 = (0,0)$ and $Z_2 = (r_1+r_2+\varepsilon, 0)$. It is easy to show that the fixed points of the combined reflections R_2R_1 and R_1R_2 are the points $(x_i, 0)$, for $i = 1, 2$, where x_1 and x_2 are the roots of the quadratic equation:

$$(r_1+r_2+\varepsilon)x^2 + \left(r_2^2 - r_1^2 - (r_1+r_2+\varepsilon)^2\right)x + r_1^2(r_1+r_2+\varepsilon) = 0.$$

Then, as ε goes to zero,

$$x_1 = r_1 - \sqrt{\frac{2r_1r_2}{r_1+r_2}}\sqrt{\varepsilon} + \mathrm{O}(\varepsilon), \qquad x_2 = r_1 + \sqrt{\frac{2r_1r_2}{r_1+r_2}}\sqrt{\varepsilon} + \mathrm{O}(\varepsilon),$$

and

$$\frac{1}{2}r_*\sqrt{\varepsilon} \leqslant |r_1 - x_j| \leqslant 2r_*\sqrt{\varepsilon}, \quad j = 1, 2. \tag{39}$$

Let $X_1 = (r_1, 0)$ and $(t_n, 0) = (R_1R_2)^n(X_1)$. We have:

$$t_{n+1} = \frac{r_1^2}{(r_1+r_2+\varepsilon) - \frac{r_2^2}{r_1+r_2+\varepsilon-t_n}},$$

and hence

$$\begin{aligned}
|t_{n+1} - x_1| &= \left|\frac{r_1^2}{(r_1+r_2+\varepsilon) - \frac{r_2^2}{r_1+r_2+\varepsilon-t_n}} - \frac{r_1^2}{(r_1+r_2+\varepsilon) - \frac{r_2^2}{r_1+r_2+\varepsilon-x_1}}\right| \\
&= |t_n - x_1|\left|\frac{r_1r_2}{(r_1+r_2+\varepsilon)(r_1+r_2+\varepsilon-t_n) - r_2^2}\right|\left|\frac{r_1r_2}{(r_1+r_2+\varepsilon)(r_1+r_2+\varepsilon-x_1) - r_2^2}\right| \\
&\leqslant \frac{|t_n - x_1|}{1 + \frac{\sqrt{\varepsilon}}{r_*}}.
\end{aligned}$$

Thus,

$$|t_n - x_1| \leqslant \frac{r_*}{4}\sqrt{\varepsilon} \quad \text{if } n \geqslant \frac{8r_*}{\sqrt{\varepsilon}}. \tag{40}$$

Observe that $|(R_1R_2)^n(X_1) - Z_2| = r_1 - t_n + r_2 + \varepsilon$. Therefore, it follows from (39) and (40) that

$$\left|(R_1R_2)^n(X_1) - Z_2\right| \geqslant r_2 + \frac{r_*}{2}\sqrt{\varepsilon} \quad \text{if } n \geqslant \frac{8r_*}{\sqrt{\varepsilon}}. \tag{41}$$

Let $X \in \overline{B}_1$. For any $0 \leqslant t \leqslant r_1$ such that $|X - Z_2| \geqslant |(t,0) - Z_2|$, we can easily see that

$$\left|R_2(X) - Z_1\right| \geqslant \left|R_2(t,0) - Z_1\right| \quad \text{and} \quad \left|R_1R_2(X) - Z_2\right| \geqslant \left|R_1R_2(t,0) - Z_2\right|.$$

Since $R_1R_2(t,0) = (s,0)$ for some s satisfying $0 \leqslant s \leqslant r_1$, then, by repeating the above inequalities, we have for each positive integer m,

$$\begin{gathered}
\left|(R_1R_2)^m(X) - Z_2\right| \geqslant \left|(R_1R_2)^m(t,0) - Z_2\right|, \\
\left|R_2(R_1R_2)^m(X) - Z_1\right| \geqslant \left|R_2(R_1R_2)^m(t,0) - Z_1\right|.
\end{gathered}$$

In particular, for the case $t = r_1$, we obtain that

$$\begin{gathered}
\left|(R_1R_2)^m(X) - Z_2\right| \geqslant \left|(R_1R_2)^m(X_1) - Z_2\right|, \\
\left|R_2(R_1R_2)^m(X) - Z_1\right| \geqslant \left|R_2(R_1R_2)^m(X_1) - Z_1\right|, \quad \forall X \in \overline{B}_1.
\end{gathered}$$

Similarly, we have:

$$\begin{gathered}\left|(R_2R_1)^m(X)-Z_1\right| \geqslant \left|(R_2R_1)^m(X_2)-Z_1\right|,\\ \left|R_1(R_2R_1)^m(X)-Z_2\right| \geqslant \left|R_1(R_2R_1)^m(X_2)-Z_2\right|, \quad \forall X\in \overline{B}_2.\end{gathered}$$

Therefore the first two inequalities in Lemma 3.1 follow from (41). The second pair of inequalities can be derived by interchanging B_1 and B_2. This completes the proof. □

Lemma 3.2. (i) *Let* $X_1=(r_1,0)$. *We have*:

$$g_2\big((R_1R_2)^m(X_1)\big),\ g_1\big((R_2R_1)^m R_2(X_1)\big) \geqslant \frac{1}{1+8(r_*/r_{\min})\sqrt{\varepsilon}}, \quad \forall m\in\mathbb{N}. \tag{42}$$

(ii) *For all* $X\in\overline{B}_1$, *we have,*

$$\begin{cases} g_2\big((R_1R_2)^m(X)\big)\leqslant 1, & \forall m\in\mathbb{N},\\ g_2\big((R_1R_2)^m(X)\big)\leqslant \dfrac{1}{1+(r_*/r_{\max})\sqrt{\varepsilon}}, & \forall m\geqslant 8r_*/\sqrt{\varepsilon}, \end{cases} \tag{43}$$

and similarly, for all $X\in\overline{B}_2$, *we have,*

$$\begin{cases} g_1\big((R_2R_1)^m(X)\big)\leqslant 1, & \forall m\in\mathbb{N},\\ g_1\big((R_2R_1)^m(X)\big)\leqslant \dfrac{1}{1+(r_*/r_{\max})\sqrt{\varepsilon}}, & \forall m\geqslant 8r_*/\sqrt{\varepsilon}. \end{cases} \tag{44}$$

Proof. Since $(R_1R_2)^m(X_1)$ is between $\widetilde{X}_1$ and $\widetilde{X}_2$ where $\widetilde{X}_1=(x_1,0)$ and $\widetilde{X}_2=(x_2,0)$ $(x_1<x_2)$ are fixed points of R_2R_1 and R_1R_2, respectively, we have:

$$\begin{aligned} g_2\big((R_1R_2)^m(X_1)\big) &= \frac{r_2^2}{|(R_1R_2)^m(X_1)-Z_2|^2} \geqslant \frac{r_2^2}{|\widetilde{X}_1-Z_2|^2}\\ &= \frac{r_2^2}{(r_1+r_2+\varepsilon-x_1)^2} \geqslant \frac{r_2^2}{(r_2+2r_*\sqrt{\varepsilon}+\varepsilon)^2} \geqslant \frac{1}{1+8(r_*/r_{\min})\sqrt{\varepsilon}}. \end{aligned}$$

The second inequality in (42) can be proved in exactly the same way. Lemma 3.1 and the definition of g_i give the upper bounds (43) and (44). □

To establish our upper bound we first observe that since $u(X)-H(X)\to 0$ as $|X|\to+\infty$, $|\nabla(u-H)|$ attains its maximum on either ∂B_1 or ∂B_2, and hence

$$\begin{aligned} \|\nabla u\|_{L^\infty(\Omega\setminus\overline{B_1\cup B_2})} &\leqslant \big\|\nabla(u-H)\big\|_{L^\infty(\Omega\setminus\overline{B_1\cup B_2})}+\|\nabla H\|_{L^\infty(\Omega)}\\ &\leqslant \big\|\nabla(u-H)|_+\big\|_{L^\infty(\partial B_1\cup\partial B_2)}+\|\nabla H\|_{L^\infty(\Omega)}\\ &\leqslant \big\|\nabla u|_+\big\|_{L^\infty(\partial B_1\cup\partial B_2)}+\|\nabla H\|_{L^\infty(\Omega)}, \end{aligned} \tag{45}$$

and

$$\|\nabla u\|_{L^\infty(B_1\cup B_2)} \leqslant \big\|\nabla u|_-\big\|_{L^\infty(\partial B_1\cup\partial B_2)}.$$

We also have from (19) and (38) that

$$\big\|\nabla u|_\pm\big\|_{L^\infty(\partial B_i)} \leqslant \left(|\lambda_i|+\frac{1}{2}\right)\|\varphi_i\|_{L^\infty(\partial B_i)}+\left\|\frac{\partial u}{\partial T}\right\|_{L^\infty(\partial B_i)}. \tag{46}$$

Let N be the first integer larger than $8r_*/\sqrt{\varepsilon}$. It then follows from (21), (43), and (44) that

$$\big|\varphi_1(X)\big| \leqslant \|\nabla H\|_{L^\infty(B_1\cup B_2)}\frac{1}{|\lambda_1|}\left(\sum_{m<N}\frac{1}{|4\lambda_1\lambda_2|^m}\left(1+\frac{1}{2|\lambda_2|}\right)+\frac{1}{|4\lambda_1\lambda_2|^N}\sum_{m=0}^{+\infty}\frac{1}{|4\lambda_1\lambda_2|^m}\left(b^{2m}+\frac{1}{2|\lambda_2|}b^{2m+1}\right)\right),$$

for any $X \in \partial B_1$, where $b := 1/(1 + (r_*/r_{\max})\sqrt{\varepsilon})$. Thus, for each $X \in \partial B_1$,

$$\begin{aligned}\left|\varphi_1(X)\right| &\leqslant \frac{C\|\nabla H\|_{L^\infty(B_1\cup B_2)}}{|\lambda_1|}\left(\frac{1}{1-|\tau|+r_*/r_{\max}\sqrt{\varepsilon}}+\frac{1-|\tau|^{8r_*/\sqrt{\varepsilon}}}{1-|\tau|}\right)\\ &\leqslant C\frac{\|\nabla H\|_{L^\infty(B_1\cup B_2)}}{|\lambda_1|(1-|\tau|+r_*/r_{\max}\sqrt{\varepsilon})},\end{aligned}$$

for any $X \in \partial B_1$. Similarly, we have the desired estimate for $\varphi_2(X)$, $X \in \partial B_2$ and hence,

$$\left\|\frac{\partial u}{\partial \nu}\right\|_{L^\infty(\partial B_1\cup\partial B_2)} \leqslant \frac{C\|\nabla H\|_{L^\infty(B_1\cup B_2)}}{1-|\tau|+r_*/r_{\max}\sqrt{\varepsilon}}. \tag{47}$$

To estimate $\partial u/\partial T$ we use Lemma 2.4. Let $\widetilde{H}$ be a harmonic conjugate of H and v be the solution to (22). Then by (47), we have:

$$\left\|\frac{\partial v}{\partial \nu}\right\|_{L^\infty(\partial B_1\cup\partial B_2)} \leqslant \frac{C\|\nabla \widetilde{H}\|_{L^\infty(B_1\cup B_2)}}{1-|\tau|+r_*/r_{\max}\sqrt{\varepsilon}}.$$

Since $\|\nabla \widetilde{H}\|_{L^\infty(B_1\cup B_2)}\|\nabla H\|_{L^\infty(B_1\cup B_2)}$, it follows from Lemma 2.4 that

$$\left\|\frac{\partial u}{\partial T}\right\|_{L^\infty(\partial B_1\cup\partial B_2)} \leqslant \frac{C\|\nabla H\|_{L^\infty(B_1\cup B_2)}}{1-|\tau|+r_*/r_{\max}\sqrt{\varepsilon}}. \tag{48}$$

Combining (45)–(48) yields the upper bound in (4) and completes the proof.

4. Proofs of Theorems 1.2 and 1.3

We suppose that $\Omega = B((0,0),\rho)$ and $B = B((\rho-r-\varepsilon,0),r)$ after rotation and translation if necessary, and that $\varepsilon \ll \rho - r$. The conductivities of Ω and B are 1 and k, $0<k\neq 1<+\infty$, respectively. Let

$$g_\Omega(X) = \frac{\rho^2}{|X|^2}, \qquad g_B(X) = \frac{r^2}{|X-(\rho-r-\varepsilon,0)|^2}.$$

The functions g_Ω and g_B play the roles of g_1 and g_2 in the previous section. Let R_Ω and R_B be the reflections with respect to Ω and B.

Lemma 4.1. *Let $X_1 = (\rho-\varepsilon,0)$, the point on $\overline{B}$ closest to $\partial\Omega$. For any positive integer n and $X \in \overline{B}$, we have*:

$$g_B\big(R_\Omega(R_BR_\Omega)^n(X)\big)g_\Omega\big((R_BR_\Omega)^n(X)\big) \leqslant g_B\big(R_\Omega(R_BR_\Omega)^n(X_1)\big)g_\Omega\big((R_BR_\Omega)^n(X_1)\big). \tag{49}$$

Proof. For any $X=(x,y)$, one can easily see that

$$\begin{aligned}g_B\big(R_\Omega(X)\big)g_\Omega(X) &= \frac{r^2}{(\frac{\rho^2x}{x^2+y^2}-(\rho-r-\varepsilon))^2+(\frac{\rho^2y}{x^2+y^2})^2}\cdot\frac{\rho^2}{x^2+y^2}\\ &= \frac{\rho^2r^2}{(\rho-r-\varepsilon)^2}\cdot\frac{1}{(\frac{\rho^2}{\rho-r-\varepsilon}-x)^2+y^2}.\end{aligned} \tag{50}$$

Since $\frac{\rho^2}{\rho-r-\varepsilon} > \rho-\varepsilon$, it immediately follows that

$$g_B\big(R_\Omega(X)\big)g_\Omega(X) \leqslant g_B\big(R_\Omega(X_1)\big)g_\Omega(X_1), \quad \forall X \in \overline{B}.$$

If X satisfies $|X-(\rho-r-\varepsilon)| \leqslant |(t,0)-(\rho-r-\varepsilon)|$, with $t > \rho-r-\varepsilon$, then

$$\left|R_BR_\Omega(X)-(\rho-r-\varepsilon)\right| \leqslant \left|R_BR_\Omega(t,0)-(\rho-r-\varepsilon)\right|.$$

Using this fact repeatedly, we have:

$$\left|(R_BR_\Omega)^n(X)-(\rho-r-\varepsilon)\right| \leqslant \left|(R_BR_\Omega)^n(X_1)-(\rho-r-\varepsilon)\right|.$$

By combining this inequality with (50), we obtain that, for $X \in \overline{B}$,

$$g_B\big(R_\Omega(R_B R_\Omega)^n(X)\big)g_\Omega\big((R_B R_\Omega)^n(X)\big) \leqslant g_B\big(R_\Omega(R_B R_\Omega)^n(X_1)\big)g_\Omega\big((R_B R_\Omega)^n(X_1)\big),$$

which completes the proof. □

Recall that P_1 and P_2 are the fixed points of the combined reflections $R_B R_\Omega$ and $R_\Omega R_B$, respectively. Observe that $P_1 \in B$. If $P_i = (x_i, 0)$ for $i = 1, 2$, then x_i $(x_1 < x_2)$ are the roots of the quadratic equation:

$$(\rho - r - \varepsilon)x^2 + \big(r^2 - \rho^2 - (\rho - r - \varepsilon)^2\big)x + \rho^2(\rho - r - \varepsilon) = 0.$$

It then follows that

$$x_1 = \rho - \sqrt{\frac{2\rho r}{\rho - r}}\sqrt{\varepsilon} + \mathrm{O}(\varepsilon), \qquad x_2 = \rho + \sqrt{\frac{2\rho r}{\rho - r}}\sqrt{\varepsilon} + \mathrm{O}(\varepsilon). \tag{51}$$

Moreover,

$$\frac{\sqrt{\varepsilon}}{r_*} \leqslant |x_j - \rho| \leqslant 2\frac{\sqrt{\varepsilon}}{r_*}, \quad j = 1, 2,$$

for ε small enough.

As a direct consequence of (49) and (51), we have the following lemma which plays a crucial role in deriving the lower bound (8).

Lemma 4.2. *For each positive integer n, the following inequality holds*:

$$g_B\big(R_\Omega(R_B R_\Omega)^n(X_1)\big)g_\Omega\big((R_B R_\Omega)^n(X_1)\big) \geqslant \frac{1}{1 + 4r^*\sqrt{\varepsilon}}. \tag{52}$$

Proof. Since $P_1 \in B$, we have

$$\begin{aligned} g_B\big(R_\Omega(R_B R_\Omega)^n(X_1)\big)g_\Omega\big((R_B R_\Omega)^n(X_1)\big) &\geqslant g_B\big(R_\Omega(R_B R_\Omega)^n(P_1)\big)g_\Omega\big((R_B R_\Omega)^n(P_1)\big) \\ &\geqslant g_B\big(R_\Omega(P_1)\big)g_\Omega(P_1). \end{aligned}$$

On the other hand, it follows from (50) and (51) that

$$g_B\big(R_\Omega(P_1)\big)g_\Omega(P_1) = \frac{\rho^2 r^2}{(\rho^2 - x_1(\rho - r - \varepsilon))^2} \geqslant \frac{1}{1 + 4r^*\sqrt{\varepsilon}}.$$

The proof is complete. □

The following lemma is also of use to us.

Lemma 4.3. *For each positive integer n, similarly to* (52) *the following holds*:

$$g_\Omega\big(R_B(R_\Omega R_B)^n(X_2)\big)g_B\big((R_\Omega R_B)^n(X_2)\big) \geqslant \frac{1}{1 + 4r^*\sqrt{\varepsilon}}. \tag{53}$$

Proof. Since the proof of Lemma 4.3 is parallel to that of Lemma 4.2, we very briefly sketch it. As before, since $P_2 \in \mathbb{R}^2 \setminus \Omega$, we can show that

$$\begin{aligned} g_B\big((R_\Omega R_B)^n(X_2)\big)g_\Omega\big((R_B R_\Omega)^n R_B(X_2)\big) &\geqslant g_B\big((R_\Omega R_B)^n(P_2)\big)g_\Omega\big((R_B R_\Omega)^n R_B(P_2)\big) \\ &= g_B(P_2)g_\Omega\big(R_B(P_2)\big). \end{aligned}$$

Since, as one can see easily, $R_\Omega(P_1) = P_2$ we have:

$$g_B(P_2)g_\Omega\big(R_B(P_2)\big) = g_B\big(R_\Omega(P_1)\big)g_\Omega(P_1),$$

and hence (53) follows. □

Next, we need the following lemma to derive the upper bound.

Lemma 4.4. *For each positive integer n,*

$$g_B\big(R_\Omega(R_B R_\Omega)^n(X)\big)g_\Omega\big((R_B R_\Omega)^n(X)\big) \leqslant 1, \quad X \in \overline{B}, \tag{54}$$

and for $n \geqslant \frac{1}{4r^*\sqrt{\varepsilon}}$,

$$g_B\big(R_\Omega(R_B R_\Omega)^n(X)\big)g_\Omega\big((R_B R_\Omega)^n(X)\big) \leqslant \frac{1}{1+r^*\sqrt{\varepsilon}}, \quad X \in \overline{B}. \tag{55}$$

Proof. The inequality (54) is obvious. We only have to prove (55). Let $(t_n, 0) := (R_B R_\Omega)^n(X_1)$. Then we have:

$$t_{n+1} = \frac{r^2}{\frac{\rho^2}{t_n} - (\rho - r - \varepsilon)} + \rho - r - \varepsilon.$$

Recall that $P_1 = (x_1, 0)$. We then have:

$$\begin{aligned}
|t_{n+1} - x_1| &= \left| \frac{r^2}{\frac{\rho^2}{t_n} - (\rho - r - \varepsilon)} - \frac{r^2}{\frac{\rho^2}{x_1} - (\rho - r - \varepsilon)} \right| \\
&= |t_n - x_1| \left[\frac{\rho r}{\rho^2 - (\rho - r - \varepsilon)x_1} \right] \left[\frac{\rho r}{\rho^2 - (\rho - r - \varepsilon)t_n} \right] \\
&\leqslant \frac{|t_n - x_1|}{1 + \sqrt{\frac{\rho - r}{\rho r}}\sqrt{\varepsilon}}.
\end{aligned}$$

If $n \geqslant \frac{1}{4r^*\sqrt{\varepsilon}}$, we get:

$$|t_n - x_1| \leqslant \frac{1}{2}\sqrt{\frac{\rho r}{\rho - r}}\sqrt{\varepsilon},$$

and therefore

$$\big|(R_B R_\Omega)^n(X_1)\big| \leqslant \rho - \sqrt{\frac{\rho r}{2(\rho - r)}}\sqrt{\varepsilon}. \tag{56}$$

By Lemma 4.1 and (50),

$$\begin{aligned}
g_B\big(R_\Omega(R_B R_\Omega)^n(X)\big)g_\Omega\big((R_B R_\Omega)^n(X)\big) &\leqslant \frac{\rho^2 r^2}{(\rho^2 - (\rho - r - \varepsilon)(\rho - \sqrt{\frac{\rho r}{2(\rho - r)}}\sqrt{\varepsilon}))^2} \\
&\leqslant \frac{1}{1 + r^*\sqrt{\varepsilon}},
\end{aligned}$$

which completes the proof. □

Since $R_B R_\Omega(X)$ is in $\overline{B}$ for any $X \in \overline{\Omega}$, we immediately obtain the following corollary:

Corollary 4.5. *For each positive integer n,*

$$g_B\big(R_\Omega(R_B R_\Omega)^n(X)\big)g_\Omega\big((R_B R_\Omega)^n(X)\big) \leqslant 1, \quad X \in \overline{\Omega},$$

and for $n > \frac{1}{4r^*\sqrt{\varepsilon}}$,

$$g_B\big(R_\Omega(R_B R_\Omega)^n(X)\big)g_\Omega\big((R_B R_\Omega)^n(X)\big) \leqslant \frac{1}{1+r^*\sqrt{\varepsilon}}, \quad X \in \overline{\Omega}.$$

We are now ready to prove Theorem 1.2 and Theorem 1.3.

Proof of Theorem 1.2. Straightforward computations yield that for $X_1 = (\rho - \varepsilon, 0)$

$$\nabla\big((R_\Omega R_B)^m \mathcal{D}_\Omega f\big)(X_1) = \prod_{n=0}^{m-1} g_B\big(R_\Omega (R_B R_\Omega)^n (X_1)\big) g_\Omega\big((R_B R_\Omega)^n (X_1)\big) \nabla \mathcal{D}_\Omega f\big((R_B R_\Omega)^m (X_1)\big). \tag{57}$$

Since $R_\Omega(P_1) = P_2$ and $R_B(P_2) = P_1$, $(R_B R_\Omega)^n(X_1)$ lies in J_1, the line segment between P_1 and X_1, for each n. We may assume,

$$\inf_{X \in J_1} \big|\big\langle \nabla \mathcal{D}_\Omega f(X), \nu_B(X_1)\big\rangle\big| \neq 0,$$

since otherwise the estimate (8) is trivial. If ε is small enough, then the length of J_1 is small and hence we may further suppose that

$$\big\langle \nabla \mathcal{D}_\Omega f(X), \nu_B(X_1)\big\rangle = \frac{\partial(\mathcal{D}_\Omega f)}{\partial x}(X)$$

has the same sign for all $X \in J_1$. It then follows from (52) and (57) that

$$\begin{aligned}\left|\frac{\partial}{\partial \nu_B}\big[(R_\Omega R_B)^m \mathcal{D}_\Omega f\big](X_1)\right| &= \left|\prod_{n=0}^{m-1} g_B\big(R_\Omega (R_B R_\Omega)^n (X_1)\big) g_\Omega\big((R_B R_\Omega)^n (X_1)\big)\big\langle \nabla \mathcal{D}_\Omega f\big((R_B R_\Omega)^m (X_1)\big), \nu_B(X_1)\big\rangle\right| \\ &\geqslant a^m \inf_{X \in J_1}\left|\frac{\partial(\mathcal{D}_\Omega f)}{\partial x}(X)\right|,\end{aligned} \tag{58}$$

where $a := (1 + 4r^*\sqrt{\varepsilon})^{-1}$.

Suppose that $k > 1$ and hence $\lambda > 0$. It follows from (58) that

$$\begin{aligned}\big|\varphi(X_1)\big| &= \left|\frac{2}{\lambda}\sum_{m=0}^{+\infty}\frac{1}{(2\lambda)^m}\frac{\partial}{\partial \nu_B}\big[(R_\Omega R_B)^m \mathcal{D}_\Omega f\big](X_1)\right| \\ &\geqslant \left|\frac{2}{\lambda}\sum_{m=0}^{+\infty}\left(\frac{a}{2\lambda}\right)^m\right| \cdot \inf_{X \in J_1}\left|\frac{\partial(\mathcal{D}_\Omega f)}{\partial x}(X)\right| \\ &\geqslant \frac{C\sigma \inf_{X \in J_1}\big|\frac{\partial(\mathcal{D}_\Omega f)}{\partial x}(X)\big|}{1 - \sigma + 4r^*\sqrt{\varepsilon}},\end{aligned}$$

since $\sigma = \frac{1}{2\lambda}$. Here C is a constant which is independent of k, r, and ε.

By (26), we have:

$$\left.\frac{\partial u}{\partial \nu_B}\right|_\pm = \left.\frac{\partial \mathcal{D}_\Omega f}{\partial \nu_B}\right|_\pm - \left.\frac{\partial \mathcal{S}_\Omega g}{\partial \nu_B}\right|_\pm + \left.\frac{\partial \mathcal{S}_B \varphi}{\partial \nu_B}\right|_\pm = \left(\lambda \pm \frac{1}{2}\right)\varphi \quad \text{on } \partial B, \tag{59}$$

and hence we obtain:

$$\left|\left.\frac{\partial u}{\partial \nu_B}\right|_+ (X_1)\right| \geqslant \frac{C \inf_{X \in J_1}\big|\frac{\partial(\mathcal{D}_\Omega f)}{\partial x}(X)\big|}{1 - \sigma + 4r^*\sqrt{\varepsilon}}. \tag{60}$$

In order to prove (9), we now estimate from below $g(X_2)$ where $X_2 = (\rho, 0)$, the point on $\partial\Omega$ which is the closest to ∂B. Elementary computations show that

$$\frac{\partial}{\partial \nu_\Omega}\big[(R_B R_\Omega)^m \mathcal{D}_\Omega f\big](X_2) = \prod_{n=0}^{m-1} g_\Omega\big(R_B (R_\Omega R_B)^n (X_2)\big) g_B\big((R_\Omega R_B)^n (X_2)\big)\big\langle \nabla \mathcal{D}_\Omega f\big((R_\Omega R_B)^m (X_2)\big), \nu_\Omega(X_2)\big\rangle.$$

Since $(R_\Omega R_B)^n(X_2)$ lies in J_2, the line between X_2 and P_2, for each n, we have as before

$$\begin{aligned}\big|g(X_2)\big| &\geqslant \left|\frac{1}{2\lambda}\sum_{m=0}^{+\infty}\left(\frac{a}{2\lambda}\right)^m + 2\right| \cdot \inf_{X \in J_2}\left|\frac{\partial(\mathcal{D}_\Omega f)}{\partial x}(X)\right| \\ &\geqslant \frac{\inf_{X \in J_2}\big|\frac{\partial(\mathcal{D}_\Omega f)}{\partial x}(X)\big|}{1 - \sigma + 4r^*\sqrt{\varepsilon}}.\end{aligned}$$

Therefore, we get:

$$\left|\frac{\partial u}{\partial \nu_\Omega}\bigg|_-(X_2)\right| \geqslant \frac{C\inf_{X\in J_2}\left|\frac{\partial(\mathcal{D}_\Omega f)}{\partial x}(X)\right|}{1-\sigma+4r^*\sqrt{\varepsilon}}. \tag{61}$$

We now prove (10). Let N be the first integer such that $N > \frac{1}{4r^*\sqrt{\varepsilon}}$. We then get from Lemma 4.4 that

$$\begin{aligned}\left|\nabla\big((R_\Omega R_B)^m\mathcal{D}_\Omega f\big)(X)\right| &\leqslant \prod_{n=0}^{m} g_B\big(R_\Omega(R_B R_\Omega)^n(X)\big)g_\Omega\big((R_B R_\Omega)^n(X)\big)\left|\nabla\mathcal{D}_\Omega f\big((R_B R_\Omega)^n(X)\big)\right| \\ &\leqslant \begin{cases}\|\nabla\mathcal{D}_\Omega f\|_{L^\infty(\overline{\Omega})} & \text{for all } m,\\ \|\nabla\mathcal{D}_\Omega f\|_{L^\infty(\overline{\Omega})}b^{m-N} & \text{if } m\geqslant N,\end{cases}\end{aligned}$$

where $b := (1+r^*\sqrt{\varepsilon})^{-1}$. It then follows that for all $X\in\overline{B}$,

$$\begin{aligned}\left|\nabla\mathcal{S}_B\varphi(X)\right| &\leqslant \frac{1}{|\lambda|}\sum_{m=0}^{+\infty}\frac{1}{(2|\lambda|)^m}\left|\nabla\big[(R_\Omega R_B)^m\mathcal{D}_\Omega f\big](X)\right| \\ &\leqslant 2\|\nabla\mathcal{D}_\Omega f\|_{L^\infty(\Omega)}\left(\sum_{m<N}\left(\frac{1}{2|\lambda|}\right)^m + \frac{1}{|2\lambda|^N}\sum_{m=0}^{+\infty}\left(\frac{b}{2|\lambda|}\right)^m\right) \\ &\leqslant C\|\nabla\mathcal{D}_\Omega f\|_{L^\infty(\Omega)}\left(\frac{1-|\sigma|^{1/(r^*\sqrt{\varepsilon})}}{1-|\sigma|}+\frac{1}{1-|\sigma|+r^*\sqrt{\varepsilon}}\right) \\ &\leqslant \frac{C\|\nabla\mathcal{D}_\Omega f\|_{L^\infty(\Omega)}}{1-|\sigma|+r^*\sqrt{\varepsilon}}.\end{aligned} \tag{62}$$

By Lemma 2.2 and (34), $|\nabla\mathcal{S}_B\varphi(X)| = g_B(X)|\nabla\mathcal{S}_B\varphi(R_B(X))|$ for $X\in\Omega\setminus\overline{B}$, and hence (62) holds for all $X\in\Omega$, i.e.,

$$\|\nabla\mathcal{S}_B\varphi\|_{L^\infty(\Omega)} \leqslant \frac{C\|\nabla\mathcal{D}_\Omega f\|_{L^\infty(\Omega)}}{1-|\sigma|+r^*\sqrt{\varepsilon}}. \tag{63}$$

Since u is harmonic in B, it follows from (59) that

$$u(X) = -2\mathcal{S}_\Omega\left(\frac{\partial u}{\partial\nu}\bigg|_-\right)(X) + \text{constant}(1-2\lambda)\mathcal{S}_B\varphi(X) + \text{constant}, \quad X\in B.$$

Since $2\lambda-1 = 2/(k-1)$, (63) gives:

$$\|\nabla u\|_{L^\infty(B)} \leqslant \frac{C\|\nabla\mathcal{D}_\Omega f\|_{L^\infty(\Omega)}}{|k-1|(1-|\sigma|+r^*\sqrt{\varepsilon})}. \tag{64}$$

By (29) and (18),

$$\mathcal{S}_\Omega g = -2\sum_{m=1}^{+\infty}\frac{1}{(2\lambda)^m}(R_\Omega R_B)^m\mathcal{D}_\Omega f - \mathcal{D}_\Omega f + \text{constant}.$$

By Corollary 4.5 and computations similar to those in (62), we obtain:

$$\|\nabla\mathcal{S}_\Omega g\|_{L^\infty(\overline{\Omega})} \leqslant \frac{C\|\nabla\mathcal{D}_\Omega f\|_{L^\infty(\overline{\Omega})}}{1-|\sigma|+r^*\sqrt{\varepsilon}}. \tag{65}$$

By combining (23), (62), and (65), we arrive at

$$\|\nabla u\|_{L^\infty(\Omega)} \leqslant C\frac{\|\nabla\mathcal{D}_\Omega f\|_{L^\infty(\Omega)}}{1-|\sigma|+r^*\sqrt{\varepsilon}}. \tag{66}$$

Since $\mathcal{D}_\Omega$ maps $\mathcal{C}^{1,\alpha}(\partial\Omega)$ into itself, it follows from the maximum principle that

$$\|\nabla\mathcal{D}_\Omega f\|_{L^\infty(\overline{\Omega})} \leqslant \|\nabla\mathcal{D}_\Omega f\|_{L^\infty(\partial\Omega)} \leqslant C\|f\|_{\mathcal{C}^{1,\alpha}(\partial\Omega)},$$

and hence we get (10). The proof of Theorem 1.2 is now complete. □

Proof of Theorem 1.3. To prove Theorem 1.3 we use Lemma 2.6. Let G be the $\mathcal{C}^{1,\alpha}$ function such that $\frac{\partial G}{\partial T} = g$ on $\partial\Omega$ and $\int_{\partial\Omega} G = 0$, and v be the solution to (31). Since $\mathcal{D}_\Omega(G)$ is a harmonic conjugate to $\mathcal{S}_\Omega g$ in Ω and $\mathbb{R}^2 \setminus \overline{\Omega}$ by Lemma 2.6, we have:

$$\left\langle \nabla\mathcal{S}_\Omega(g)(X), T_B(X_1)\right\rangle = -\left\langle \nabla\mathcal{D}_\Omega(G)(X), \nu_B(X_1)\right\rangle,$$

and

$$\left\langle \nabla\mathcal{S}_\Omega(g)(X), T_\Omega(X_2)\right\rangle = -\left\langle \nabla\mathcal{D}_\Omega(G)(X), \nu_\Omega(X_2)\right\rangle, \quad X \in \mathbb{R}^2 \setminus \partial\Omega.$$

Thus (11) and (12) follow from (8), (9), and (32).

As one can see in the proof of Lemma 2.6, v is a harmonic conjugate to u in $\Omega \setminus \overline{B}$ and $\frac{1}{k}v$ is a harmonic conjugate to u in B. Therefore by (66) we get:

$$\|\nabla u\|_{L^\infty(\Omega\setminus\overline{B})} \leqslant \|\nabla v\|_{L^\infty(\Omega)} \leqslant \frac{C\|\nabla\mathcal{S}_\Omega g\|_{L^\infty(\Omega)}}{1 - |\sigma| + r^*\sqrt{\varepsilon}}. \tag{67}$$

Moreover, by (64), we have:

$$\begin{aligned}\|\nabla u\|_{L^\infty(B)} &\leqslant \frac{1}{k}\|\nabla v\|_{L^\infty(B)} \leqslant \frac{1}{k}\min\left\{\frac{1}{|\frac{1}{k} - 1|}, 1\right\}\frac{C\|\nabla\mathcal{S}_\Omega g\|_{L^\infty(\Omega)}}{1 - |\sigma| + r^*\sqrt{\varepsilon}} \\ &\leqslant \frac{C\|\nabla\mathcal{S}_\Omega g\|_{L^\infty(\Omega)}}{1 - |\sigma| + r^*\sqrt{\varepsilon}}.\end{aligned}$$

Since $\|\nabla\mathcal{S}_\Omega g\|_{L^\infty(\Omega)} \leqslant C\|g\|_{\mathcal{C}^\alpha(\partial\Omega)}$, we finally get (13). This completes the proof of Theorem 1.3. □

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